

Exact phase diagrams for an Ising model on a two-layer Bethe lattice

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Using an iteration technique, we obtain exact expressions for the free energy and the magnetization of an Ising model on a two-layer Bethe lattice with intralayer coupling constants J_1 and J_2 for the first and the second layer, respectively, and interlayer coupling constant J_3 between the two layers; the Ising spins also couple with external magnetic fields, which are different in the two layers. We obtain exact phase diagrams for the system and find that when $|J_3| \rightarrow 0$, $\Delta T_c \equiv [T_c(J_3) - T_c(0)]/T_c(0) \sim |J_3|/J_1|^{1/\psi}$, where $T_c(J_3)$ is the phase-transition temperature for the system with interlayer coupling constant J_3 and the shift exponent ψ is 1 for $J_1 = J_2$ and is 0.5 for $J_1 \neq J_2$. Such results are consistent with predictions of a scaling theory. We also derive equations for ΔT_c when $|J_3|$ approaches ∞ . [S1063-651X(99)08606-7]

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I. INTRODUCTION

The physical properties of various magnetic-layered structures and superlattices have been intensely studied both experimentally and theoretically for reasons ranging from fundamental investigations of phase transitions to technical problems encountered in thin-film magnets [1]. Experimentally, submonolayer and monolayer films of ferromagnetic materials offer challenging opportunities to fabricate materials with various novel magnetic properties, such as giant magnetoresistance, surface magnetic anisotropy, enhanced surface magnetic moment, and surface magnetoelastic coupling. On theoretical grounds, surface magnetism has been treated within several different frameworks: mean-field approximations [2], effective-field theories [3], spin-fluctuation theory [4], renormalization-group methods [5], two-site cluster approximations [6], and Monte Carlo techniques [7]. Though each method has its own advantages, they all have limitations in treating film systems. Numerical techniques such as the Monte Carlo method can provide very accurate results for properties of finite systems; however, they are computation intensive and can be carried out only for relatively small system sizes.

Since exact solutions for realistic layer systems on regular lattices are generally unavailable, one relies on approximation schemes to obtain a qualitative picture of the phase diagram. In our approach, we replace the original two-layer regular lattice by a two-layer Bethe lattice with the same coordination number q as the original lattice. Once this approximation is made, we can solve our model exactly as in the case of one-layer Bethe lattice [8]. It is now widely recognized (see, e.g., [9–12]) that in many cases solutions of a spin model on Bethe or generalized Bethe lattices are qualitatively better approximations for the regular lattice than so-

lutions obtained by conventional mean-field theories, because of the presence of correlations (albeit weak ones) in the former [13] and the lack of correlations in the latter. It has also been found that phase diagrams of an Ising model on a Husimi tree (a Bethe-like lattice) with two-spin and three-spin interactions [10,11] closely match exact phase diagrams of an Ising model on a two-dimensional Kagome lattice with two-spin and three-spin interactions [14]. Of course, our approximation also has limitations: since correlations are weak [13], it predicts a transition temperature that is higher than that for a regular lattice, and it is usually not reliable for predicting critical exponents. On the other hand, the Monte Carlo method will be highly reliable for predicting critical exponents. But we believe that our approach correctly gives the general shape of the phase diagram.

It is well known that the Bethe and Bethe-like lattices cannot be embedded in a finite-dimensional Euclidean space without distortion in their bond angles and lengths [15]. On the other hand, it has been pointed out by Mosseri and Sadoc [16] that these structures can be considered as regular lattices of fixed bond angles and lengths if they are embedded in a two-dimensional space of constant negative curvature (the hyperbolic or Lobachevsky plane **H2** [17]). The surface of negative curvature are now being introduced to describe some complex structure with large cells, formed by inorganic and organic materials, which can be considered as crystals of surface and films. Among them are the cubic crystalline structures formed by amphiphilic molecules in the presence of water [18,19] and a magnetically coupled three-dimensional (Terephthalato) manganese (II) network [20,21]. Furthermore, non-Euclidean hyperbolic symmetries have even been found in hexagonal and cubic-close-packed Euclidean crystals [22]. The local structural similarities between that material and negatively curved Bethe lattice suggest that the two-layer Bethe lattice considered in the present paper can be related to some physical systems.

In this paper, we use an iteration technique to obtain exact expressions for the free energy and the magnetization of an Ising model on a two-layer Bethe lattice with intralayer cou-

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pling constants J_1 and J_2 for the first and the second layer, respectively, and interlayer coupling constant J_3 between two layers; the Ising spins also couple with external magnetic fields, which are different in the two layers. Allowing for a difference in these two fields is important, because they act in an opposing manner on the zero-field boundaries [23]. We obtain exact phase diagrams for the system and find that when $|J_3| \rightarrow 0$, $\Delta T_c \equiv [T_c(J_3) - T_c(0)]/T_c(0) \sim |J_3|/J_1^{1/\psi}$, where $T_c(J_3)$ is the phase-transition temperature for the system with interlayer coupling constant J_3 and the shift exponent ψ is 1 for $J_1 = J_2$ and is 0.5 for $J_1 \neq J_2$. Such results are consistent with predictions of a scaling theory [24–26]. We also derive equations for ΔT_c when $|J_3|$ approach ∞ .

The theoretical works on thin-layer systems before the present paper were less general. The system of coupled two-dimensional Ising planes on regular lattices, e.g., the square lattice, is not exactly soluble; however, it has been investigated by a variety of approximate methods. Ballentine [28] used high-temperature series expansions to study the model with $J_1 = J_2 = J_3$ [27]. This work was later extended by Allan [29] to films up to five layers and by Capehart and Fisher [30] to films up to ten layers. The two-layer system where the interlayer coupling constant differs from the intralayer coupling constant was studied by Abe [24] in the context of a scaling theory valid in the limit of a weak interlayer coupling. The more general case in which $J_1 \neq J_2$ has also received some attention. The most complete treatment was that of Oitmaa and Enting [31], who combined mean-field theory, scaling theory, and high-temperature expansions in a study of the two-layer model, and calculated the variation of the critical temperature, the layer magnetizations, and the interlayer correlation function with J_3 . Recently, Ferrenberg and Landau [32] considered the same two-layer problem using Monte Carlo simulations and mean-field theory. The Bethe lattice version of a thin film was first studied in [33], in which the phase diagrams of coupled bilayers with $J_1 = J_2$ and zero external fields were obtained.

More recently, there have been many investigations of the two-layer Ising model due to stimulation from experiments for ultrathin magnetic films [26,34–43]. The experimental study of such systems has made significant advances in recent years due to improvements in surface-force apparatus and microscopic techniques. Many theoretical studies were devoted to the model with two exchange parameters, i.e., J_s for spins on free surfaces and J for all other spins. However, in our opinion, the exchange interaction (J_3) between the surface and the second layer has an important influence on the surface magnetic order. Therefore, in the present paper, we study an Ising model on a two-layer Bethe lattice with three exchange parameters J_1 , J_2 , and J_3 and in the presence of magnetic fields, which are different in the two layers. We will calculate the free energy of that system and investigate its critical properties.

In previous papers [44,45], the relations between the free energies of a spin-1 Ising model on Bethe and Cayley trees and of a multisite Ising model on Husimi lattices and generalized Cayley trees are obtained. In that approach one obtains the free-energy from recursion relations, and equations for physical quantities by differentiation of the free-energy functionals with respect to external fields. Recently, an elegant and original method of computing the bulk free energy

for any model on infinite Bethe and Husimi lattices was presented by Gujrati [12]. In the present paper, we extend this method to study an Ising model on a bilayer Bethe lattice. In particular, we will calculate exact phase diagrams and the shift exponent ψ for the system.

In 1970-1971, Abe [24] and Suzuki [25] used a scaling theory to predict that the shift exponent ψ for the two-layer planar lattice Ising model is equal to the susceptibility exponent ($\gamma = 1.75$) of the two-dimensional Ising model. In 1992, Angelini *et al.* predicted that $\psi = 1.5$ [40]; in 1993, Lipowski and Suzuki predicted that $\psi = 2$ [36]. Very recently, Horiguchi and Tsushima [26] used a β -function approach [46] to obtain $\psi = 1.73 \pm 0.04$ for the two-layer square lattice Ising model with $J_1 = J_2$, which is very close to the theoretically predicted value $\psi = \gamma = 1.75$. They also found that the shift exponent ψ for the system with $J_1 \neq J_2$ is 0.5 and explained this value in terms of a scaling theory. In 1998 Lipowski [47] used a transfer-matrix mean-field approximation to calculate the shift exponent ψ . He found $\psi = 1.79$ for the system with $J_1 = J_2$ and $\psi = 0.501$ for the system with $J_1 \neq J_2$.

In this paper, we consider both $J_1 = J_2$ and $J_1 \neq J_2$ for the Ising model on a two-layer Bethe lattice and obtain exact values of ψ , which are consistent with the predictions of the scaling theory.

The outline of this paper is as follows. In Sec. II, we present the bilayer Ising model and discuss different types of ground states. In Sec. III, we derive the exact free energy, equations of state, and order parameters for the Ising model on the bilayer Bethe lattice. In Sec. IV, we investigate temperature dependence of the order parameters and discuss the phase diagrams. In Sec. V, we calculate the critical temperature in weak and strong vertical coupling regimes and obtain exactly the shift exponent ψ . In Sec. VI, we conclude with some general remarks concerning our results.

II. TWO-LAYER MODEL AND ITS GROUND STATES

In this section, we consider an ultrathin film composed of two atomic layers G_1 and G_2 such that the lattice structures of G_1 and G_2 are identical and each of them have N sites and coordination number q ; the corresponding lattice sites in G_1 and G_2 are labeled by i and i' , respectively, where $1 \leq i, i' \leq N$ and they are nearest neighbors (NN) to each other. The Ising Hamiltonian on such two-layer lattice can be written as

$$-\beta H = J_1 \sum_{\langle ij \rangle} s_i s_j + J_2 \sum_{\langle i'j' \rangle} \sigma_{i'} \sigma_{j'} + J_3 \sum_{\langle ii' \rangle} s_i \sigma_{i'} + h_1 \sum_i s_i + h_2 \sum_{i'} \sigma_{i'}, \quad (1)$$

where $\beta = (k_B T)^{-1}$ with k_B being the Boltzmann constant and T being the temperature, s_i and $\sigma_{i'}$ take values ± 1 , J_1 and J_2 are, respectively, the coupling constants of the exchange interaction between the pair of NN spins in the first and the second layer, J_3 is the coupling constant between a spin in the first layer and its NN in the second layer, and h_1 and h_2 are magnetic fields acting on spins in the first and the second layer, respectively.

This model has three order parameters, two of which correspond to the thermal average of total spins of the first and the second layers, respectively,

$$m_1 = \frac{1}{N} \sum_{i=1}^N \langle s_i \rangle, \quad m_2 = \frac{1}{N} \sum_{i'=1}^N \langle \sigma_{i'} \rangle. \quad (2)$$

These order parameters can be defined by variation of the partition function with respect to h_1 and h_2 . The total magnetization density m and the density of staggered magnetization η are defined by

$$m = \frac{1}{2}(m_1 + m_2), \quad \eta = \frac{1}{2}(m_1 - m_2). \quad (3)$$

The third order parameter corresponds to the interlayer spin-spin correlation function between NN spins of adjacent layers,

$$\rho = \frac{1}{N} \sum_{i=i'=1}^N (\langle s_i \sigma_{i'} \rangle - \langle s_i \rangle \langle \sigma_{i'} \rangle). \quad (4)$$

Before studying the temperature dependences of the order parameters, let us investigate the ground states of the model at $T=0$ analytically. The ground-state energy in units of $|J_1|$ and in the absence of magnetic fields may be described by the following Hamiltonian:

$$E = - \sum_{\langle plaq \rangle} \left[\frac{J_1}{|J_1|} s_i s_j + \frac{J_2}{|J_1|} \sigma_{i'} \sigma_{j'} + \frac{J_3}{q|J_1|} (s_i \sigma_{i'} + s_j \sigma_{j'}) \right]. \quad (5)$$

Here the summation goes over all plaquettes and each plaquette consists of four nearest-neighbor pairs of the two-layer system with one pair, $\langle ij \rangle$, on G_1 , one pair, $\langle i'j' \rangle$, on G_2 , and two pairs, $\langle ii' \rangle$ and $\langle jj' \rangle$, connecting G_1 and G_2 .

By comparing the values of E for different spin configurations, we obtain the ground-state phase diagrams shown in Figs. 1(a) and 1(b) for $J_1 > 0$ and $J_1 < 0$, respectively. We find five types of ground states with the following values of the order parameters (m, η, ρ) :

- (I) $m = \pm 1, \quad \eta = 0, \quad \rho = 0,$
- (II) $m = 0, \quad \eta = \pm 1, \quad \rho = 0,$
- (III) $m = 0, \quad \eta = 0, \quad \rho = 1,$
- (IV) $m = 0, \quad \eta = 0, \quad \rho = -1,$
- (V) $m = \pm 1/2, \quad \eta = \pm 1/2, \quad \rho = 0.$

The coordinates $(J_2/|J_1|, J_3/q|J_1|)$ of the multiphase points are

$$\begin{aligned} A_1 &\rightarrow (0,0), & B_1 &\rightarrow (-1,1), \\ C_1 &\rightarrow (-1,-1) & \text{for } J_1 > 0 \end{aligned} \quad (6)$$

and

$$\begin{aligned} A_2 &\rightarrow (0,0), & B_2 &\rightarrow (1,1), & C_2 &\rightarrow (1,-1) \\ & & & & \text{for } J_1 < 0. \end{aligned} \quad (7)$$

Phase (I) represents the usual ferromagnetic ordering $m_1 = m_2$ ($\eta = 0$). Phase (II) represents ferromagnetic ordering in G_1 and G_2 , but magnetizations in G_1 and G_2 are antiparallel, i.e., $m_1 = -m_2$ and $m = 0$ (interlayer ordering is antiferromagnetic type). It is worthwhile to note that this phase corresponds to the well-known compensation phenomenon, which occurs when the magnetizations of two layers cancel each other instead of being equal. Phase (III) represents the antiferromagnetic ordering in both layers ($m_1 = m_2 = 0$) where interlayer ordering is ferromagnetic ($\rho = 1$). Phase (IV) represents the totally antiferromagnetic ordering ($\rho = -1$). Phase (V) represents the ferromagnetic ordering ($m = \pm 1/2, \eta = \pm 1/2$), which is equivalent to the case that the ground state of one layer is ferromagnetic and the ground state of another layer is antiferromagnetic. The phases (I)–(V) will be referred to as (F) ferromagnetic, (C) compensated, (M) mixed, (A) antiferromagnetic and surface ferromagnetic (SF) phase, respectively.

III. EQUATIONS OF STATE AND FREE ENERGY

Let us consider an Ising model on a bilayer Bethe lattice, which is constructed by connecting to the central pair of sites q pairs in order to form the first generation and by connecting successively $(q-1)$ pairs to each pair in a generation to form the next generation. The result is an infinite lattice in

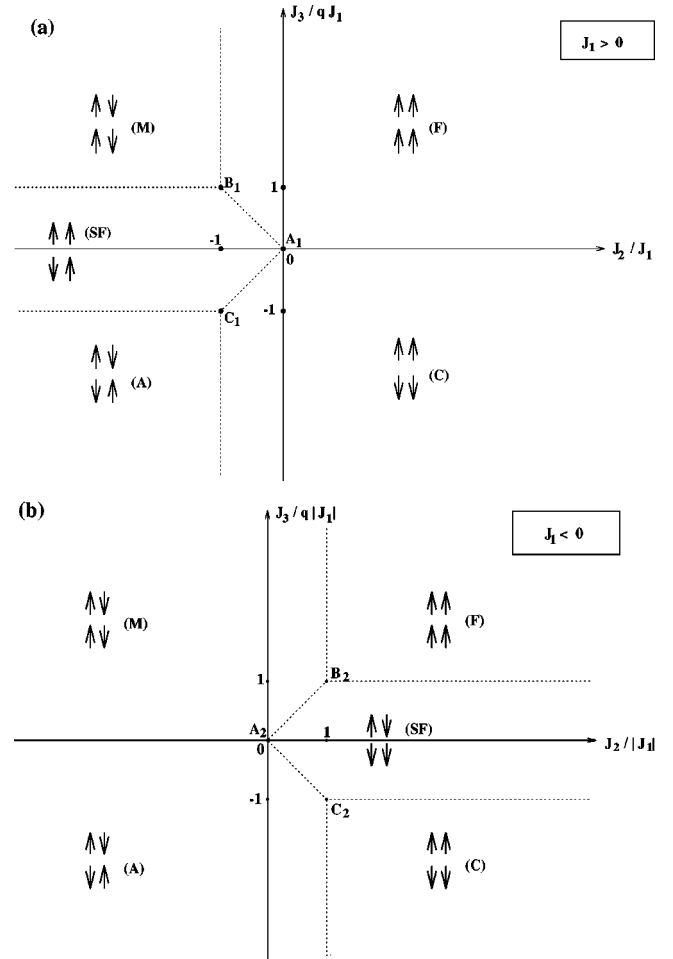


FIG. 1. Ground-state phase diagram of the two-layer Ising model for (a) $J_1 > 0$ and (b) $J_1 < 0$.

which every site has $(q+1)$ nearest neighbors, where q nearest neighbors are in the same layer as the site and one nearest neighbor is in another layer.

The partition function of the system represented by Eq. (1) may be written as

$$Z = \sum_{\{\sigma, s\}} \exp \left\{ J_1 \sum_{\langle ij \rangle} s_i s_j + J_2 \sum_{\langle i' j' \rangle} \sigma_{i'} \sigma_{j'} + J_3 \sum_{\langle ii' \rangle} \sigma_{i'} s_i + h_1 \sum_i s_i + h_2 \sum_{i'} \sigma_{i'} \right\}, \quad (8)$$

where the sum goes over all configurations of the system.

Now we derive exact recursion relations for Z . When the Bethe tree is cut apart at the central pair, it separates into q identical branches, each of which contains $(q-1)$ branches. The partition function can be written as follows:

$$Z = \sum_{\{\sigma_0, s_0\}} \exp \{ J_3 \sigma_0 s_0 + h_1 s_0 + h_2 \sigma_0 \} g_n^q(\sigma_0, s_0), \quad (9)$$

where σ_0 and s_0 are the spins of the central pair, n is the number of generations ($n \rightarrow \infty$ corresponds to the thermody-

amic limit where surface effects may be neglected), and $g_n(\sigma_0, s_0)$ is the partition function of a separate branch. Each branch, in turn, can be cut apart at the pair of sites nearest to the central pair. The expression for $g_n(\sigma_0, s_0)$ can, therefore, be written in the following form:

$$g_n(\sigma_0, s_0) = \sum_{\{\sigma_1, s_1\}} \exp \{ J_1 s_0 s_1 + J_2 \sigma_0 \sigma_1 + J_3 \sigma_1 s_1 + h_1 s_1 + h_2 \sigma_1 \} g_{n-1}^{-1}(\sigma_1, s_1). \quad (10)$$

Let us introduce the following variables x_n , y_n , and t_n ,

$$x_n = \frac{g_n(+ +)}{g_n(- -)}, \quad y_n = \frac{g_n(+ -)}{g_n(- +)}, \quad t_n = \frac{g_n(- +)}{g_n(- -)}.$$

From Eq. (10) we easily obtain the recursion relations:

$$\begin{aligned} x_n &= f_1(x_{n-1}, y_{n-1}, t_{n-1}), \\ y_n &= f_2(x_{n-1}, y_{n-1}, t_{n-1}), \\ t_n &= f_3(x_{n-1}, y_{n-1}, t_{n-1}), \end{aligned} \quad (11)$$

where

$$\begin{aligned} f_1(x_n, y_n, t_n) &= \frac{A_n \exp(J_1 + J_2) + \exp(-J_1 - J_2) + D_n \exp(J_1 - J_2) + B_n \exp(-J_1 + J_2)}{A_n \exp(-J_1 - J_2) + \exp(J_1 + J_2) + D_n \exp(-J_1 + J_2) + B_n \exp(J_1 - J_2)}, \\ f_2(x_n, y_n, t_n) &= \frac{A_n \exp(-J_1 + J_2) + \exp(J_1 - J_2) + D_n \exp(-J_1 - J_2) + B_n \exp(J_1 + J_2)}{A_n \exp(J_1 - J_2) + \exp(-J_1 + J_2) + D_n \exp(J_1 + J_2) + B_n \exp(-J_1 - J_2)}, \\ f_3(x_n, y_n, t_n) &= \frac{A_n \exp(J_1 - J_2) + \exp(-J_1 + J_2) + D_n \exp(J_1 + J_2) + B_n \exp(-J_1 - J_2)}{A_n \exp(-J_1 - J_2) + \exp(J_1 + J_2) + D_n \exp(-J_1 + J_2) + B_n \exp(J_1 - J_2)}, \end{aligned}$$

with

$$A_n = x_n^{q-1} \exp(2h_1 + 2h_2), \quad D_n = t_n^{q-1} \exp(-2J_3 + 2h_1), \quad B_n = y_n^{q-1} t_n^{q-1} \exp(-2J_3 + 2h_2).$$

Through x_n , y_n , and t_n , one can express the magnetization and other thermodynamic quantities, so we can say that in the thermodynamic limit ($n \rightarrow \infty$) x_n , y_n , and t_n determine the states of the system. For this reason the recursion relations can also be called the equations of state for the two-layer Ising model. The magnetizations of the first and the second layers as well as the spin-spin correlation function between spins of adjacent layers are expressed by

$$m_1 = \langle s_0 \rangle = \frac{x_n A_n - 1 + t_n D_n - y_n t_n B_n}{x_n A_n + 1 + t_n D_n + y_n t_n B_n}, \quad (12)$$

$$m_2 = \langle \sigma_0 \rangle = \frac{x_n A_n - 1 - t_n D_n + y_n t_n B_n}{x_n A_n + 1 + t_n D_n + y_n t_n B_n}, \quad (13)$$

$$\langle \sigma_0 s_0 \rangle = \frac{x_n A_n + 1 - t_n D_n - y_n t_n B_n}{x_n A_n + 1 + t_n D_n + y_n t_n B_n}. \quad (14)$$

We are interested in the case when (x_n, y_n, t_n) converges to a stable point (x_s, y_s, t_s) , which is associated with the thermodynamic solutions of the two-layer Ising model. In this case the recursion relations (or equations of state) given by Eq. (11) can be rewritten in the following form:

$$\left(\frac{1-y}{1+y} \right)^{q-1} \exp(2h_2 - 2h_1) = \frac{u_1 - v_1}{u_1 + v_1}, \quad (15)$$

$$\left(\frac{1-t}{1+t} \right)^{q-1} \exp(2h_1 + 2h_2) = \frac{u_2 - v_2}{u_2 + v_2}, \quad (16)$$

$$x^{2(q-1)} \left(\frac{1-y^2}{1-t^2} \right)^{q-1} \exp(-4J_3) = \frac{u_1^2 - v_1^2}{u_2^2 - v_2^2}, \quad (17)$$

where

$$u_1 = c_1 x - c_2, \quad v_1 = s_1 x y + s_2 t,$$

$$c_1 = \cosh(J_1 + J_2), \quad s_1 = \sinh(J_1 + J_2),$$

$$u_2 = c_1 - c_2 x, \quad v_2 = s_1 t + s_2 x y,$$

$$c_2 = \cosh(J_1 - J_2), \quad s_2 = \sinh(J_1 - J_2),$$

and

$$x = \frac{1 + y_s}{1 + x_s} t_s, \quad y = \frac{1 - y_s}{1 + y_s}, \quad t = \frac{1 - x_s}{1 + x_s}.$$

The total magnetization density [$m = (m_1 + m_2)/2$], the density of the staggered magnetization [$\eta = (m_1 - m_2)/2$] and the density of the interlayer spin-spin correlation function ($\rho = \langle \sigma_0 s_0 \rangle - m_1 m_2$) can be expressed as

$$m = -\frac{tu_2 + v_2}{u_2 + tv_2 + xu_1 + v_1 xy}, \quad (18)$$

$$\eta = \frac{(xyu_1 + xv_1)}{u_2 + tv_2 + xu_1 + v_1 xy}, \quad (19)$$

$$\rho = \frac{(1-t^2)(u_2^2 - v_2^2) - x^2(1-y^2)(u_1^2 - v_1^2)}{(u_2 + tv_2 + xu_1 + xyv_1)^2}. \quad (20)$$

In the case when (x_n, y_n, t_n) converge to a stable point (x_s, y_s, t_s) , we can obtain an equation for the free-energy functional F :

$$\begin{aligned} -\beta F = & -\frac{1}{8} \ln(u_1^2 - v_1^2)(u_2^2 - v_2^2) \\ & + \frac{q-1}{8} \ln x^2(1-t^2)(1-y^2) \\ & - \frac{q-2}{4} \ln(u_2 + tv_2 + xu_1 + xyv_1) \\ & + \frac{1}{2} \ln 2 + \frac{q}{4} \ln |c_1^2 - c_2^2|. \end{aligned} \quad (21)$$

In deriving this equation we have used the exact relation between the free energy of the Bethe lattice and Cayley trees [12,44,45].

It is easily seen that the expressions for the order parameters m_1 , m_2 , and ρ can be obtained by differentiation of the free-energy functional of Eq. (21) with respect to the magnetic fields h_1 , h_2 and the coupling constant J_3 , respectively. In this sense, the interlayer coupling constant J_3 is analogous to an external field.

This result for the free energy is very useful for locating phase transitions in case of multiple solutions of the equation of state and for determining the equilibrium state. Using this free-energy functional one can obtain the full phase diagram, describing not only the continuous phase transitions but also the discontinuous ones.

In the next section we will discuss the critical properties of our model; in particular, we will calculate the critical temperature as a function of ratios of coupling constants and will show the full phase diagram in the three-dimensional param-

eter space spanned by coupling constants J_1 , J_2 , and J_3 for different values of the coordination number q .

IV. PHASE DIAGRAMS

Now we consider the critical properties of the Ising model on a two-layer Bethe lattice with different ferromagnetic coupling constants ($J_1 > 0, J_2 > 0$). Without loss of generality, we need only consider $J_1 \geq J_2$. The phase transition occurs when $h_1 = h_2 = 0$. In this case Eqs. (15), (16), and (17) become

$$x^{2(q-1)} \left(\frac{1-y^2}{1-t^2} \right)^{q-1} \exp(-4J_3) = \frac{u_1^2 - v_1^2}{u_2^2 - v_2^2}, \quad (22)$$

$$(1+y)^{q-1}(u_1 - v_1) = (1-y)^{q-1}(u_1 + v_1) \Leftrightarrow v_1 = y u_1 \Phi(y^2), \quad (23)$$

$$(1+t)^{q-1}(u_2 - v_2) = (1-t)^{q-1}(u_2 + v_2) \Leftrightarrow v_2 = t u_2 \Phi(t^2), \quad (24)$$

where

$$\Phi(x) = \left(\sum_{n=0}^{[q-2/2]} C_{2n+1}^{q-1} x^n \right) / \left(\sum_{n=0}^{[q-1/2]} C_{2n}^{q-1} x^n \right), \quad (25)$$

$$\Phi(0) = q - 1,$$

and C_k^{q-1} is a binomial coefficient.

It can be seen that there always exists one solution of this system:

$$(i) \quad y = t = 0, \quad x^{q-1} \exp(-2J_3) = \frac{u_1}{u_2}.$$

This solution corresponds to the high-temperature paramagnetic phase ($m_1 = 0, m_2 = 0$). In addition to the first solution, the equations of state also have other solutions with $y \neq 0, t \neq 0$:

$$(ii) \quad \begin{aligned} s_2^2 x &= [u_1 \Phi(y^2) - s_1 x][u_2 \Phi(t^2) - s_1], \\ t^2 [u_2 \Phi(t^2) - s_1] &= x y^2 [u_1 \Phi(y^2) - s_1 x], \end{aligned}$$

$$x^{2(q-1)} \left(\frac{1-y^2}{1-t^2} \right)^{q-1} \exp(-4J_3) = \frac{u_1^2 - v_1^2}{u_2^2 - v_2^2}.$$

Of course, only the solution, which minimizes the free-energy functional (21), is thermodynamically stable. The others correspond to unstable or metastable states. If there are two or more solutions that have the same minimum free energies, these phases coexist and the system has a first-order phase transition.

When these two solutions merge into one solution, i.e., (i) = (ii), we obtain the critical line of second-order phase transitions,

$$\exp(2J_3) = \frac{c_1 - c_2 x_\lambda}{c_1 x_\lambda - c_2} x_\lambda^{q-1}, \quad (26)$$

where x_λ is the solution of the following equation:

$$s_2^2 x_\lambda = \{[(q-1)c_1 - s_1]x_\lambda - (q-1)c_2\} \\ \times [(q-1)c_1 - s_1 - (q-1)c_2 x_\lambda]. \quad (27)$$

It is convenient now to introduce the parameters k_1 and k_2 ,

$$k_1 = \sqrt{\tanh J_1 \tanh J_2}, \\ k_2 = \sqrt{\tanh\left(-J_1 + 0.5 \ln \frac{q}{q-2}\right) \tanh\left(-J_2 + 0.5 \ln \frac{q}{q-2}\right)}. \quad (28)$$

The two solutions $x_\lambda^{(1,2)}$ of Eq. (27) can thus be expressed as

$$x_\lambda^{(1)} = \frac{1 - k_1 k_2}{1 + k_1 k_2}, \quad x_\lambda^{(2)} = \frac{1 + k_1 k_2}{1 - k_1 k_2}, \quad (29)$$

and the corresponding expressions for the λ lines of the second-order phase transition in the three-dimensional parameter space spanned by J_1 , J_2 , and J_3 will take the form,

$$\exp(2J_3^{(1)}) = \frac{k_1 + k_2}{k_1 - k_2} \left(\frac{1 - k_1 k_2}{1 + k_1 k_2} \right)^{q-1}, \quad J_3 > 0 \quad (30)$$

$$\exp(2J_3^{(2)}) = \frac{k_1 - k_2}{k_1 + k_2} \left(\frac{1 + k_1 k_2}{1 - k_1 k_2} \right)^{q-1}, \quad J_3 < 0. \quad (31)$$

The critical lines of the second-order phase transition given by Eqs. (30) and (31) separate the paramagnetic (P) phase from the ferromagnetic (F) and compensated (C) phases, which are in turn separated by a first-order phase transition line.

Before discussing the phase diagram, it is convenient to introduce the parameters $n = J_2/J_1$, $\delta = J_3/qJ_1$, and $T = J_1^{-1}$. In terms of the T , n , and δ , Eqs. (30) and (31) for the λ lines imply a relation $T = T_c(n, \delta)$, which locates the critical temperature as a function of n and δ for arbitrary values of the coordination number q . The two critical lines start at

$$T_c^{max} = \frac{2(1+n)}{\ln[q/(q-2)]}, \quad |J_3| \rightarrow \infty, \quad (32)$$

and meet each other at

$$T_c^{min} = \frac{2}{\ln[q/(q-2)]}, \quad J_3 = 0. \quad (33)$$

At $J_3 = 0$ the system has a second critical point,

$$T_c^{sec} = \frac{2n}{\ln[q/(q-2)]}, \quad J_3 = 0. \quad (34)$$

The phase diagrams (T_c versus δ) of the Ising model on the two-layer Bethe lattice for different values $n = 1, 0.75, 0.5, 0.25, 0.1$, and for different values of coordination number $q = 3, 4, 6, \infty$ are shown in Figs. 2(a), 2(b), and 2(c). A few comments are in order. For $J_3 = 0$ we recover

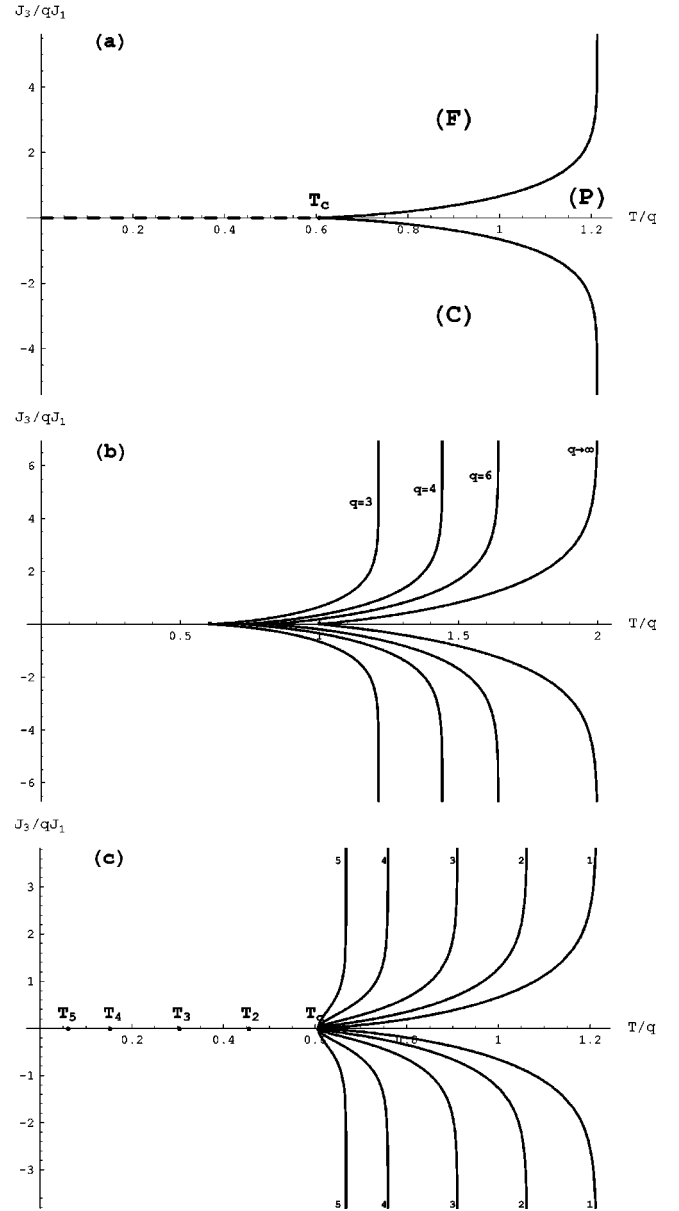


FIG. 2. Phase diagram on $(J_3/qJ_1, T/q)$ plane for an Ising model on a two-layer Bethe lattice with intralayer coupling constants J_1 and J_2 for the first and the second layer, respectively, and interlayer coupling constant J_3 between two layers; here, q is the coordination number for one-layer Bethe lattice and $T = 1/J_1 > 0$. (a) $J_1 = J_2$ and $q = 3$. A first-order phase boundary (dashed line) separates two ordered phases designated by (F) and (C). The solid line denotes the second-order phase-transition line, which separates paramagnetic phase (P) from two ordered phases (F) and (C). Notice that T_c is the critical temperature of the Ising model on one-layer Bethe lattice. (b) $J_1 = J_2$ and $q = 3, 4, 6$, and ∞ . (c) $q = 3$ and $n = J_2/J_1 = 1, 0.75, 0.5, 0.25$, and 0.1 , which are denoted on curves by 1, 2, 3, 4, and 5, respectively. T_2, T_3, T_4 , and T_5 are the second critical point of Eq. (34) for curves labeled by 2, 3, 4, and 5, respectively.

two critical temperatures of two single-layer Bethe lattices with different intralayer ferromagnetic coupling constants (J_1 and J_2). In the opposite limit of $|J_3| \rightarrow \infty$, the critical temperature goes asymptotically to a value given by Eq. (32) with the effective intralayer coupling constant $J_1(1+n)$, since the interlayer pairs become rigidly correlated.

V. WEAK AND STRONG INTERLAYER COUPLING REGIMES

The spin-1/2 Ising model on a two-layer square lattice is exactly soluble only in the cases $J_3=0$ and $|J_3|\rightarrow\infty$, where it is related to the one-layer square Ising model. When $J_3=0$, the Hamiltonian given by Eq. (1) describes two uncoupled Ising modes or, equivalently, two free fermionic fields. In strong vertical interaction limits $|J_3|\rightarrow\infty$, each pair of spins coupled across the layers will act as a single spin, and the Hamiltonian given by Eq. (1) describes a one-layer Ising model with (J_1+J_2) as the coupling constant.

In the weak interlayer coupling regime ($J_3\rightarrow 0$) the shift exponent ψ can be defined by

$$\Delta T_c \equiv \frac{T_c(J_3) - T_c(0)}{T_c(0)} \sim |J_3/J_1|^{1/\psi}, \quad (35)$$

where $T_c(J_3)$ is the critical temperature when the system has interlayer coupling constant J_3 .

In this section we calculate exactly the shift exponent for the Ising model on a two-layer Bethe lattice. In the weak-coupling regime we obtain

$$\Delta T_c = b_1(q) \frac{|J_3|}{J_1} \quad \text{for } J_1 = J_2 \quad (n=1) \quad (36)$$

and

$$\Delta T_c = b_2(q, n) \left(\frac{J_3}{J_1} \right)^2 \quad \text{for } J_1 > J_2 \quad (n < 1), \quad (37)$$

where

$$b_1(q) = \frac{1}{q-2}, \quad b_2(q, n) = \frac{\ln a}{8(q-1)} \left(\frac{1+a^{n-1}}{1-a^{n-1}} \right) (a^{2n}-1),$$

with $a = q/(q-2)$.

Thus we find that the shift exponent ψ for the system with $J_1 = J_2$ is equal to 1, which coincides with theoretically predicted results $\psi = \gamma = 1$ for the single-layer Bethe lattice. For the system with $J_1 \neq J_2$, we find that $\psi = 0.5$, which also exactly coincides with the value predicted by the scaling theory [26].

In the strong coupling regime we have

$$\frac{T_c(J_3)}{T_c^{\max}} = 1 - K \exp\left(-\frac{2|J_3/J_1|}{T_c^{\max}}\right) \quad \text{for } J_1 \geq J_2 \quad (n \leq 1), \quad (38)$$

where

$$K = \frac{2(q-1)}{\ln[q/(q-2)]} (1-b^2)b^{q-2}$$

with

$$b = \frac{q-2}{2(q-1)} \left[\left(\frac{q}{q-2} \right)^{1/(n+1)} + \left(\frac{q}{q-2} \right)^{n/(n+1)} \right].$$

It is easy to see from Eqs. (36), (37), and (38) that the behavior of the strong-coupling expansions is very different from the behavior in the weak-coupling regime. It seems that we have obtained Eq. (38) for the two-layer system with different intralayer coupling constants ($J_1 \neq J_2$). It should be noted that for the case $J_1 = J_2$, equations similar to Eq. (38) had been obtained by approximate methods [36,40].

VI. SUMMARY AND DISCUSSION

In the present paper we have investigated an Ising model on a bilayer Bethe lattice with intralayer coupling constants J_1 and J_2 for the first and the second layers, respectively, and interlayer coupling constant J_3 between the two layers. We first analyze phase diagrams of ground states, then using an iteration technique to obtain exact expressions for order parameters and the free energy of the bilayer Ising model [Eqs. (18)–(21)]. We then obtain exact phase diagrams of Eqs. (30) and (31) and analyze these equations in the weak and strong interlayer coupling regimes, see Eqs. (36)–(38). The shift exponents ψ in Eqs. (36) and (37) are the first exact result to support the scaling theory for ψ , that states that ψ is equal to the exponent of magnetic susceptibility for $J_1 = J_2$ and is equal to 0.5 for $J_1 \neq J_2$ [24–26]. To the best of our knowledge, it seems that Eq. (38) is a new result.

In Sec. II, we present very rich phase diagrams for ground states. However, in Sec. IV we consider only phase diagrams for $J_1 \geq J_2 > 0$. It is of interest to study the evolution of phase diagrams in Sec. II as the temperature increases from 0 to high temperatures. However, the analysis of such general phase diagrams is quite complicated.

The dependence of various quantities on the film thickness is a topic of current interest. In principle, we can extend our calculations from two layers to n layers. For such a n -layer system, we can introduce n external magnetic fields h_1, h_2, \dots, h_n (h_i for the i th layer with $1 \leq i \leq n$), C_2^n ($=n(n-1)/2!$) interlayer coupling constants for two-layer coupling, C_3^n interlayer coupling constants for three-layer coupling, \dots , and C_n^n ($=1$) coupling constant for n -layer coupling. Therefore, the total number of such *field* coupling parameters are $C_1^n + C_2^n + C_3^n + \dots + C_n^n = 2^n - 1$. Equations (15)–(17) for two-layer systems for three *field* coupling parameters (h_1, h_2, J_3) can be extended to $2^n - 1$ equations for $2^n - 1$ *field* coupling parameters. It is very difficult to find analytic or numerical solutions of these equations for $n > 2$. However, we can simplify the problem by reducing the number of independent *field* coupling parameters, i.e., setting $h_i = h$ for $1 \leq i \leq n$ and keeping only a two-layer interlayer coupling parameter for two nearest-neighbor layers. We are working in this direction.

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